Hybrid Laplace Transform/Finite-Element Method for Two-Dimensional Transient Heat Conduction

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A powerful method of analysis, the combined use of the Laplace transform and the finite-element method, is applicable to the problem of time-dependent heat flow systems. The present method removes the time terms using the Laplace transform and then solves the associated equation with the finite-element technique. The associated temperature transform is inverted by the method of Dubner and Abate for obtaining the required results. The present results are compared in tables to the analytical solutions and other numerical data, and it is found that the present solutions are stable and convergent to them. There exists no time step; thus, the present method is a useful tool in solving lengthy problems or in other engineering applications.

Nomenclature

$[B]$	= gradient matrix
c	= specific heat, J/Kg·K
CON	= free parameter, $= vT$
$[D]$	= material property matrix
e	= element number
E	= total number of elements
$\{f^{(e)}\}$	= element force vector
$\{F\}$	= global force vector
h	= heat-transfer coefficient
$\{k^{(e)}\}$	= element conduction matrix
$[K]$	= global conduction matrix
K_x	= thermal conductivity in the x direction, W/m·K
K_y	= thermal conductivity in the y direction, W/m·K
L_x	= length of domain in the x direction, m
L_y	= length of domain in the y direction, m
M	= finite number of terms of the infinite series
n	= node number
$[N^{(e)}]$	= element shape function matrix
q	= specified surface heat flow, W/m^2
Q	= heat generated within the material, W/m^3
r	= number of nodes assigned to element (e)
R	= real domain
s	= complex number, $= v + iw$
t	= time
T	= half-period of $f(t)$
v, w	= parameter of the complex number s
x, y	= coordinate
α	= thermal diffusivity of the material
$\{\tilde{\theta}\}$	= global nodal temperature vector
θ	= temperature
$\tilde{\theta}$	= associated temperature transform
θ_0	= initial temperature
θ_s	= boundary temperature
θ_∞	= surrounding temperature
$\tilde{\theta}_i$	= discrete nodal temperature
ρ	= density
ℓ_x	= direction cosine in the x direction
ℓ_y	= direction cosine in the y direction

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Introduction

SEVERAL different techniques of numerical analysis have been presented for solving transient heat conduction problems such as the finite-difference, finite-element, and boundary integral equation methods.[1-14] The finite-element method based on a variational principal was used to analyze the unsteady problem of heat transfer by Gurtin.[4] Emery and Carson[5] and Visser[6] applied variational formulations in their finite-element solutions to nonstationary temperature distribution problems. In 1974, Bruch and Zyvoloski[7] used the finite-element weighted residual process to solve transient linear and nonlinear two-dimensional heat conduction problems. Rizzo and Shippy[9] and Liggett and Liu[10] used the boundary integral equation method to solve transient linear one-dimensional problems. Recently, Rources and Alarcon[11] and Cheng and Liggett[12] extended this method to solve transient linear two-dimensional problems. For the finite-difference method, Bhattacharya[13] and Lick,[14] to name a few, applied the improved finite-difference method to time-dependent heat conduction problems.

The best advantage of previous methods is their ability to handle a wide variety of multidimensional, composite structures with the irregular geometry and time-dependent energy generation. Despite their advantage, they also have practical limitations. First, their major drawback is that it is often necessary to take very small time steps to avoid undesirable numerically induced oscillations[15-18] in the solution. A severe limitation on the time step may require an excessive amount of computer time. In lengthy solutions, this fact is especially discouraging. Second, all the interior temperatures must be computed at each time step, even though one may be interested only in a single temperature or heat flow as an output. In previous works, this required the solution of n simultaneous algebraic equations at each time step when the usual Crank-Nicolson algorithm was used. On the other hand, these approaches tend to increase cost when the solutions must be carried out over long time periods.

The present method, the combined use of the Laplace transform and the finite-element method, is used to solve two-dimensional transient linear heat conduction problems. To show the numerical accuracy and efficiency of the present method, four different examples are analyzed. It is found from these examples that the present method can overcome the disadvantages of previous works. In addition, the present results also agree with the corresponding exact solutions or those of Bruch and Zyvoloski[7] and Bell.[8] However, the primary purpose of the present study is to extend the present method to other two- or three-dimensional linear time-dependent problems with irregular geometry and multiple substances.

Mathematical Analysis

Governing Equation

The governing differential equation for two-dimensional transient heat conduction in a homogeneous and isotropic solid body is given by

$$K_x\frac{\partial^2\theta}{\partial x^2}+K_y\frac{\partial^2\theta}{\partial y^2}+Q=\rho c\frac{\partial\theta}{\partial t} \tag{1}$$

with boundary conditions of the following types:

$$\theta=\theta_s \qquad \text{on } S_1 \tag{2}$$

$$-K_x\frac{\partial\theta}{\partial x}\ell_x-K_y\frac{\partial\theta}{\partial y}\ell_y=q \qquad \text{on } S_2 \tag{3}$$

$$-K_x\frac{\partial\theta}{\partial x}\ell_x-K_y\frac{\partial\theta}{\partial y}\ell_y=h(\theta-\theta_\infty) \qquad \text{on } S_3 \tag{4}$$

and the initial condition

$$\theta(x,y,0)=\theta_0(x,y) \tag{5}$$

where the union of the surface S_1, S_2, and S_3 forms the complete boundary of the solid having an area A. The part of the boundary on which the temperature θ is prescribed is S_1, the part of the boundary on which the heat flux is prescribed is S_2, and the part of the boundary on which the convective heat transfer is prescribed is S_3.[21,22]

In order to remove time dependence from the governing equation and boundary conditions, the scheme of the Laplace transform will be utilized. The Laplace transform of a real function f: $R\to R$ with $f(t)=0$ for $t<0$ and its inversion formulas are defined as

$$F(s)=L[f(t)]=\int_0^\infty e^{-st}f(t)\mathrm{d}t \tag{6}$$

$$f(t)=L^{-1}[F(s)]=\frac{1}{2\pi i}\int_{v-i\infty}^{v+i\infty}e^{st}F(s)\mathrm{d}s \tag{7}$$

with $s=v+iw$; v, $w\in R$.

Though $v\in R$ is arbitrary, it must be greater than the real parts of all the singularities of $F(s)$. The integrals in Eqs. (6) and (7) exist for $Re(s)>a\in R$ if 1) f is locally integrable, 2) there exist $t_0\geq 0$ and k, $a\in R$, such that

$$|f(t)|\leq ke^{at} \text{ for all } t\geq t_0 \tag{8}$$

3) for all $t\in(0,\infty)$ there is a neighborhood in which $f(t)$ is of bounded variation.

It is assumed that $f(t)$ must satisfy Eq. (8) and that there are no singularities of $F(s)$ to the right of the origin. Therefore, Eqs. (6) and (7) are defined for all $v>0$. The possibility of choosing $v>0$ arbitrarily is the basis of the methods of Dubner and Abate[19] and Durbin.[20] It should be noted that a good choice of the free parameters M and $CON=vT$ is not only important for the accuracy of the results but also for the acceleration of convergence. These problems have been considered in the present study. In addition, the "optimal" v for fixed M and T can also be approximately determined by the present method.

Formulation of Associated Finite-Element Equations

After taking the Laplace transform with respect to time, Eqs. (1-4) become

$$K_x\frac{\partial^2\tilde\theta}{\partial x^2}+K_y\frac{\partial^2\tilde\theta}{\partial y^2}+\tilde Q=\rho c(s\tilde\theta-\theta_0) \tag{9}$$

with boundary conditions

$$\tilde\theta=\frac{\theta_s}{s} \quad \text{on } S_1 \tag{10}$$

$$K_x\frac{\partial\tilde\theta}{\partial x}\ell_x+K_y\frac{\partial\tilde\theta}{\partial y}\ell_y+\tilde q=0 \quad \text{on } S_2 \tag{11}$$

$$K_x\frac{\partial\tilde\theta}{\partial x}\ell_x+K_y\frac{\partial\tilde\theta}{\partial y}\ell_y+h\left(\tilde\theta-\frac{\theta_\infty}{s}\right)=0 \quad \text{on } S_3 \tag{12}$$

The functional formulation equivalent to Eq. (9) and its boundary conditions, Eqs. (10-12), can be written as

$$I=\frac{1}{2}\iint_A\left[K_x\left(\frac{\partial\tilde\theta}{\partial x}\right)^2+K_y\left(\frac{\partial\tilde\theta}{\partial y}\right)^2-2\tilde Q\tilde\theta+\rho cs\tilde\theta^2-2\rho c\theta_0\tilde\theta\right]\mathrm{d}x\mathrm{d}y+\int_{S_2}\tilde q\tilde\theta\mathrm{d}s+\frac{h}{2}\int_{S_3}\left(\tilde\theta-\frac{\theta_\infty}{s}\right)^2\mathrm{d}s \tag{13}$$

The functional I can be written as a sum of individual functionals defined for all elements of the assemblage, i.e.,

$$I=\sum_{e=1}^{E}I^{(e)} \tag{14}$$

Suppose that domain A is divided into simplex triangular elements and that the distribution of $\tilde\theta$ within each element is assumed to be

$$\tilde\theta^{(e)}(x,y)=\sum_{i=1}^{r}N_i^{(e)}(x,y)\tilde\theta_i=[N^{(e)}]\{\tilde\theta\} \tag{15}$$

where r is the number of nodes assigned to element (e) and $\tilde\theta_i$ are the discrete nodal temperatures. In the process of minimization, the following matrices will be defined:

$$[B^{(e)}]=\begin{bmatrix}\dfrac{\partial N_i^{(e)}}{\partial x} & \dfrac{\partial N_j^{(e)}}{\partial x}\\ \dfrac{\partial N_i^{(e)}}{\partial y} & \dfrac{\partial N_j^{(e)}}{\partial y}\end{bmatrix} \tag{16}$$

$$[D^{(e)}]=\begin{bmatrix}K_x^{(e)} & 0\\ 0 & K_y^{(e)}\end{bmatrix} \tag{17}$$

$$\{g^{(e)}\}=[B^{(e)}]\{\tilde\theta\} \tag{18}$$

$$\{g^{(e)}\}^T=\left[\frac{\partial\tilde\theta^{(e)}}{\partial x}\ \frac{\partial\tilde\theta^{(e)}}{\partial y}\right] \tag{19}$$

By substituting Eqs. (15-19) into Eq. (13), the discretized functional $I^{(e)}$ for one element can be expressed in terms of the nodal temperatures:

$$I^{(e)}=\frac{1}{2}\iint_{A^{(e)}}\{\tilde\theta\}^T[B^{(e)}]^T[D^{(e)}][B^{(e)}]\{\tilde\theta\}\mathrm{d}x\mathrm{d}y-\iint_{A^{(e)}}\left(\tilde Q-\frac{1}{2}\rho cs\{\tilde\theta\}^T[N^{(e)}]^T+\rho c\theta_0\right)[N^{(e)}]\{\tilde\theta\}\mathrm{d}x\mathrm{d}y+\int_{S_2^{(e)}}\tilde q[N^{(e)}]\{\tilde\theta\}\mathrm{d}s+\frac{h}{2}\int_{S_3^{(e)}}\left(\{\tilde\theta\}^T[N^{(e)}]^T[N^{(e)}]^T\{\tilde\theta\}-\frac{2}{s}\theta_\infty[N^{(e)}]\{\tilde\theta\}+\frac{1}{s^2}\theta_\infty^2\right)\mathrm{d}s \tag{20}$$

The minimization of the functional I occurs when

$$\frac{\partial I}{\partial\{\tilde{\theta}\}}=\frac{\partial}{\partial\{\tilde{\theta}\}}\sum_{e=1}^{E} I^{(e)}=\sum_{e=1}^{E}\frac{\partial I^{(e)}}{\partial\{\tilde{\theta}\}}=0 \tag{21}$$

Differentiating Eq. (20) with respect to $\{\tilde{\theta}\}$ yields

$$\begin{aligned}\frac{\partial I^{(e)}}{\partial\{\tilde{\theta}\}}=&\left(\iint_{A^{(e)}}[B^{(e)}]^T[D^{(e)}][B^{(e)}]\mathrm{d}x\mathrm{d}y\right.\\&+\iint_{A_3^{(e)}}\rho c s[N^{(e)}]^T[N^{(e)}]\mathrm{d}x\mathrm{d}y\\&+\left.\int_{S_3^{(e)}}h[N^{(e)}]^T[N^{(e)}]\mathrm{d}s\right)\{\tilde{\theta}\}\\&-\iint_{A^{(e)}}\tilde{Q}[N^{(e)}]^T\mathrm{d}x\mathrm{d}y+\int_{S_2^{(e)}}\tilde{q}[N^{(e)}]^T\mathrm{d}s\\&-\int_{S_3^{(e)}}\frac{h}{s}\theta_\infty[N^{(e)}]^T\mathrm{d}s-\iint_{A^{(e)}}\rho c\theta_0[N^{(e)}]^T\mathrm{d}x\mathrm{d}y\end{aligned} \tag{22}$$

After the rearrangement, Eq. (22) can be rewritten in the simplified form as

$$\frac{\partial I^{(e)}}{\partial\{\tilde{\theta}\}}=[K^{(e)}]\{\tilde{\theta}\}+\{f^{(e)}\} \tag{23}$$

where

$$\begin{aligned}[K^{(e)}]=&\iint_{A^{(e)}}[B^{(e)}]^T[D^{(e)}][B^{(e)}]\mathrm{d}x\mathrm{d}y\\&+\iint_{A^{(e)}}\rho c s[N^{(e)}]^T[N^{(e)}]\mathrm{d}x\mathrm{d}y\\&+\int_{S_3^{(e)}}h[N^{(e)}]^T[N^{(e)}]\mathrm{d}s\end{aligned} \tag{24}$$

$$\begin{aligned}\{f^{(e)}\}=&-\iint_{A^{(e)}}(\tilde{Q}+\rho c\theta_0)[N^{(e)}]^T\mathrm{d}x\mathrm{d}y\\&+\int_{S_2^{(e)}}\tilde{q}[N^{(e)}]^T\mathrm{d}s-\int_{S_3^{(e)}}\frac{h}{s}\theta_\infty[N^{(e)}]^T\mathrm{d}s\end{aligned} \tag{25}$$

Substituting Eq. (23) into Eq. (21) yields

$$\sum_{e=1}^{E}([K^{(e)}]\{\tilde{\theta}\}+\{f^{(e)}\})=0 \tag{26}$$

Rewrite Eq. (26) as

$$[K]\{\tilde{\theta}\}=\{F\} \tag{27}$$

where

$$[K]=\sum_{e=1}^{E}[K^{(e)}]$$

is an $n\times n$ band matrix with complex number, $\{\tilde{\theta}\}$ is an $n\times 1$ vector representing the unknown temperatures,
and

$$\{F\}=-\sum_{e=1}^{E}\{f^{(e)}\}$$

is an $n\times 1$ vector representing the unknown forcing terms.

Numerical Processes

The numerical inversion form of the Laplace transform can be written in the development of trigonometric integrals, i.e.,

$$F(s)=\int_0^\infty e^{-vt}f(t)[\cos(wt)-i\sin(wt)]\mathrm{d}t \tag{28}$$

or

$$F(s)=Re\{F(v+iw)\}+i\,Im\{F(v+iw)\} \tag{29}$$

Expanding Eq. (7) with $\mathrm{d}s=i\mathrm{d}w$ yields

$$\begin{aligned}f(t)=&\frac{1}{2\pi i}\int_{-\infty}^{\infty}e^{vt}[\cos(wt)+i\sin(wt)][Re\{F(s)\}\\&+i\,Im\{F(s)\}]i\mathrm{d}w=\frac{e^{vt}}{2\pi}\left[\int_{-\infty}^{\infty}(Re\{F(s)\}\cos(wt)\right.\\&-Im\{F(s)\}\sin(wt))\mathrm{d}w\\&+\left.i\int_{-\infty}^{\infty}(Im\{F(s)\}\cos(wt)+Re\{F(s)\}\sin(wt))\mathrm{d}w\right]\end{aligned} \tag{30}$$

Rewrite Eq. (30) as

$$\begin{aligned}f(t)=&\frac{e^{vt}}{2\pi}\left[\int_{-\infty}^{\infty}\int_0^\infty e^{-v\tau}f(\tau)\cos(w(\tau-t))\mathrm{d}\tau\mathrm{d}w\right.\\&-\left.i\int_{-\infty}^{\infty}\int_0^\infty e^{-v\tau}f(\tau)\sin(w(\tau-t))\right]\mathrm{d}\tau\mathrm{d}w\end{aligned} \tag{31}$$

In Eq. (31), $\sin(w(\tau-t))$ is an odd function of w, so that the second integral is always zero. This implies that $f(t)$ can be written as

$$\begin{aligned}f(t)=&\frac{e^{vt}}{\pi}\left[\int_0^\infty(Re\{F(s)\}\cos(wt)\right.\\&-\left.Im\{F(s)\}\sin(wt))\mathrm{d}w\right]\end{aligned} \tag{32}$$

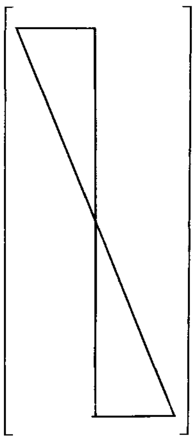
In accordance with the method of Durbin,[20] the Fourier series expansion of $H(t)=e^{-vt}f(t)$ in the interval $[0, 2T]$ can be derived as follows:

$$\begin{aligned}f(t)=&\frac{e^{vt}}{T}\left[\frac{1}{2}Re\{F(v)\}\right.\\&+\sum_{k=0}^{\infty}Re\left\{F\left(v+i\frac{k\pi}{T}\right)\right\}\cos\left(\frac{k\pi}{T}t\right)\\&-\left.\sum_{k=0}^{\infty}Im\left\{F\left(v+i\frac{k\pi}{T}\right)\right\}\sin\left(\frac{k\pi}{T}t\right)\right]-F1(v,t,T)\\&\text{for } 0<t<2T\end{aligned} \tag{33}$$

where $F1(v,t,T)$ is the discretization error given by

$$F1(v,t,T)=\sum_{k=1}^{\infty}e^{-2kvT}f(2kT+t) \tag{34}$$

Suppose that the solution located at a specific node in the given domain is required. Then the double-direct Gaussian elimination algorithm and the method of Dubner and Abate[19] can be used to solve Eq. (27). This treatment is different from that of previous works, which must compute all nodal values for each time step until the specific time is reached. The present method can compute the specific nodal temperature at a specific time. In addition, it is obvious that the present method takes less computer time if lengthly solutions are required. The procedure of the double-direct Gaussian elimination algorithm is that the band matrix in Eq. (27) is forwardly eliminated to the specific position and then backwardly eliminated to this position. Finally, the result where $\tilde{\theta}_k$ is equal to g_k is obtained. The result of the double-direct Gaussian elimination algorithm is expressed as follows:



$$\begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{\theta}_k \\ \cdot \\ \cdot \\ \cdot \\ \tilde{\theta}_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_k \\ \cdot \\ \cdot \\ \cdot \\ g_n \end{bmatrix} \tag{35}$$

Numerical Application

Example 1

This example analyzes the heat dissipation of a square plate shown in Fig. 1. Its initial temperature is equal to 30 K. The temperatures on all the boundaries are kept constant. For simplicity, $K_x = K_y = 1.25$ W/(m·K), $L_x = L_y = 3.0$ m, $\Delta x = \Delta y = 0.3$ m, and $\rho c = 1$ $J/(\mathrm{m}^3 \cdot \mathrm{K})$ are also assumed. Thus, the governing equation, boundary conditions, and initial condition for this example can be given by

$$K_x \frac{\partial^2 \theta}{\partial x^2} + K_y \frac{\partial^2 \theta}{\partial y^2} = \frac{\partial \theta}{\partial t}$$

$$0 \le L_x \le 3.0, \quad 0 \le L_y \le 3.0, \qquad t > 0 \tag{36}$$

$$\theta(0,y,t) = \theta(L_x,y,t) = \theta(x,0,t) = \theta(x,L_y,0) = 0 \tag{37}$$

and

$$\theta(x,y,0) = 30 \tag{38}$$

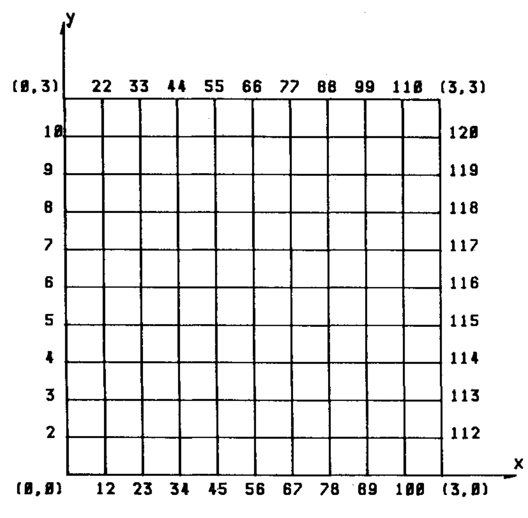


Fig. 1 Location of nodes for the first and second examples.

where L_x and L_y are the lengths of the solution domain in the x and y directions, respectively.

$$\theta(x,y,t) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} An \sin\left(\frac{n\pi x}{L_x}\right) \sin \times \left(\frac{j\pi y}{L_y}\right) \exp\left[-\left(\frac{k_x n^2}{L_x^2} + \frac{k_y j^2}{L_y^2}\right)\pi^2 t\right] \tag{39}$$

where

$$An = \frac{120}{nj\pi^2}[(-1)^n - 1][(-1)^j - 1] \tag{40}$$

The nodal temperatures at $t = 1.2$ s when 121 node modeling is used are listed in Table 1. Table 1 also shows a comparison of the present solutions to the corresponding exact solutions and those of Bruch and Zyvoloski.[7] It can be seen from Table 1 that the present solutions are not only in agreement with the corresponding exact solutions but are also more accurate than those of Bruch and Zyvoloski.[7] Obtaining a nodal temperature of this example takes about 40 s CPU-time (CDC Cyber 180) for the present method.

Example 2

The governing equation and initial condition are assumed to be similar to Eqs. (36) and (38). However, boundary conditions are considered as

$$\theta(x,0,t) = \theta(L_x,y,t) = \theta(x,L_y,t) = 0 \tag{41}$$

and

$$\frac{\partial \theta}{\partial x}(0,y,t) = 0 \tag{42}$$

The exact solution for this example is given by

$$\theta(x,y,t) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} Bn \cos\left(\frac{(2n-1)\pi x}{2L_x}\right) \times \sin\left(\frac{j\pi y}{L_y}\right) \exp\left[-\left(\frac{k_x(2n-1)^2}{4L_x^2} + \frac{k_y j^2}{L_y^2}\right)\pi^2 t\right] \tag{43}$$

Table 1 Solutions of example 1 corresponding to t = 1.2 s

Node position	Present solution	Exact solution	Ref. 7 $\Delta t = 0.05$
13	0.166	0.173	0.185
21	0.166	0.173	0.185
28	0.998	1.065	0.139
37	1.116	1.186	1.268
41	1.116	1.186	1.268
60	1.614	1.723	1.843
61	1.697	1.812	1.938
80	0.813	0.862	0.922
81	1.116	1.186	1.268

Table 2 Solutions of example 2 corresponding to $t = 1.2$ s

Node position	Present solution	Exact solution	Ref. 7 $\Delta t = 0.05$
24	1.788	1.815	1.865
28	6.018	5.872	6.036
32	1.788	1.815	1.865
37	4.580	4.476	4.603
41	4.580	4.476	4.603
61	4.492	4.455	4.589
86	1.697	1.705	1.760
94	1.981	1.985	2.057

where

$$Bn = \frac{240}{\pi^2 j(2n-1)}(-1)^{n+2}[(-1)^j - 1] \tag{44}$$

Table 2 shows a comparison of the present solutions to the corresponding exact solutions and those of Bruch and Zyvoloski.[7] It can be seen that the present solutions are more accurate than those of Ref. 7 when the same space model (121 nodes) is used. Obtaining a nodal temperature of this example also takes about 40 s CPU time (CDC Cyber 180).

Example 3

This example considers the problem with sharp corners in the boundary shown in Fig. 2. The governing equation is given by

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{\partial\theta}{\partial t} \tag{45}$$

with boundary conditions

$$\theta(0,y,t) = 1000 \tag{46}$$

$$\theta(1,y,t) = 0 \tag{47}$$

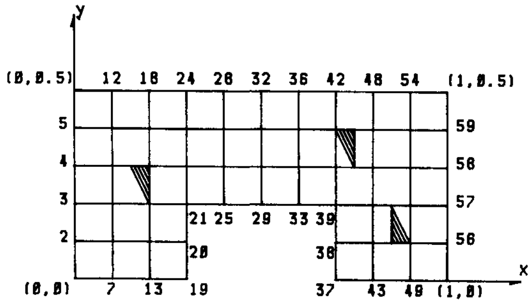


Fig. 2 Location of nodes for the third and fourth examples.

and $\partial\theta/\partial n = 0$ on all other boundaries, where $\partial/\partial n$ is the derivative normal to the boundary. The initial condition is a small time solution in a plane medium and is taken to be

$$\theta(x,y,0) = \mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right) \tag{48}$$

where $t = 0.0005$.

The comparison of the present solutions ($\Delta x = \Delta y = 0.1$ and $\Delta x = 0.05$, $\Delta y = 0.1$) to those of Bruch and Zyvoloski[7] and Bell[8] ($\Delta x = \Delta y = 0.05$ and $\Delta t = 0.0005$) is listed in Table 3. It can be seen from Table 3 that the present solutions agree with those of Bruch and Zyvoloski[7] and Bell,[8] though the larger value of Δx is chosen. Table 3 also shows that the present solutions obtained by the smaller Δx are more convergent to other numerical data. There exists no time step; thus, the selection of Δx or Δy for the present method is independent of that of Δt. However, to obtain the more accurate solution, the smaller time increment must be required in the techniques of Bruch and Zyvoloski[7] and Bell.[8] In the present study, the CPU time of running 60–90 time steps for the methods of Bruch and Zyvoloski[7] and Bell[8] is equal to that of one run for the present method when the same element size is used. However, it can be found from the third example that the methods of Bruch and Zyvoloski[7] and Bell[8] must run 200 times in order to obtain solutions at $t = 0.1$ s. In the third example, obtaining a nodal temperature takes about 80 s CPU time (CDC Cyber 180) for the present method. This implies that the present method takes less computer time than the methods of Bruch and Zyvoloski[7] and Bell[8] when the lengthy solutions or a specific nodal temperature are required.

Example 4

The fourth example analyzes the problem with the heat surface source Q, where Q is equal to 500/(shaded area) W/m^3. Suppose that this example has the same conditions and geometry as the third example shown in Fig. 2 except with shaded surface sources. Thus, the governing equation for this example is given by

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + Q = \frac{\partial\theta}{\partial t} \tag{49}$$

Table 3 Solutions of example 3 corresponding to $t = 0.1$ s

	Present solution		Ref. 7	Ref. 8	
Node position	$\Delta x = 0.1$, $\Delta y = 0.1$	$\Delta x = 0.05$, $\Delta y = 0.1$	$\Delta x = 0.05$, $\Delta y = 0.05$, $\Delta t = 0.005$	5-term approx. / 6-term approx.	Simple explicit $\Delta x = \Delta y = 0.05$, $\Delta t = 0.0005$
12	843.158	842.411	842	842 / 842	842
15	749.968	750.347	755	756 / 758	749
17	696.677	694.664	692	694 / 694	692
22	575.876	571.990	566	568 / 566	565
30	294.933	293.311	289	291 / 290	290
37	33.478	30.655	28	29 / 27	31
39	87.213	87.159	86	88 / 86	91
43	25.676	24.639	23	24 / 23	25
46	60.001	60.178	59	60 / 59	61
51	21.217	21.112	20	21 / 20	21
54	30.576	30.497	30	30	30

Table 4 Nodal temperatures with heat surface sources at $t = 0.1$ s for $\Delta x = 0.05$ m and $\Delta y = 0.1$ m

Node position	Present solution
9	883.618
12	851.574
15	775.394
30	318.560
37	58.530
39	159.149
46	90.548
51	41.314
54	44.552
(0.25,0.4)	640.478

For this example, neither analytical solutions nor numerical data can be used for comparison to the present solutions. However, it can be seen from the results of the first three examples that the present results should be accurate. Furthermore, it can be seen from Tables 3 and 4 that the nodal temperatures near the heat surface source are raised. This physical tendency is reasonable. In the fourth example, it also takes about 80 s CPU time (CDC Cyber 180) to obtain a nodal temperature.

Discussion and Conclusions

The present method involving the combined use of the Laplace transform and the finite-element method is applicable to transient linear two-dimensional heat conduction problems. To illustrate the numerical accuracy and efficiency of the present method, four different examples have been analyzed. It is found that the results of the first three examples are stable and convergent to the corresponding exact solutions or other numerial data. In addition, less computer time is expended in solving lengthy problems.

There exists no time step; thus, the present method can compute the specific position temperature at a specific time. This treatment is different from that of previous works, which must compute all the nodal temperatures at each time step until the required time is reached. Accordingly, it can be said that the present technique is a powerful tool when lengthy solutions or a specific nodal temperature are required. Recently, the idea of the Laplace transform combining the boundary integral equation method[9-12] has been applicable to transient linear problem. However, the major drawback of this method is that it must find the fundamental solution of the corresponding problem in advance. Therefore, this method is not very powerful for solving transient linear problems. It can be found from the present analysis that the present method does not have this limitation.

The present method can be readily applicable to problems with time-dependent boundary conditions. Results of these developments will be published in a future paper.

Many more difference equations corresponding to the heat equation can be derived by the present method than are illustrated here. However, these results should give an indication of the basic procedure and also of some of the more useful results possible by this procedure. The procedure is general, consistent, and accurate. The present study applies the procedure to heat conduction problems, but the procedure as described should be generally applicable to most parabolic partial-differential equations.

References

[1]Carnahan, B., Luther, H.A., and Wilkes, J.O., "Applied Numerical Method," Wiley, New York, 1969, pp. 454–461.

[2]Wilson, E.L. and Nickell, R.E., "Application of the Finite Element Method to Heat Convection Analysis," *Nuclear Engineering and Design,* North Holland, Amsterdam, 1966, pp. 276–286.

[3]Carslaw, H.S. and Jaeger, J.C., "Conduction of Heat in Solids," 2nd ed., Clarendon Press, London, 1959, pp. 466–478.

[4]Gurtin, M.E., "Variational Principles for Linear Initial-Value Problem," *Quarterly of Applied Mathematics,* Vol. 22, Oct. 1964, pp. 252–256.

[5]Emery, A.F. and Carson, W.W., "An Evaluation of the Use of the Finite Element Method in the Computation of Temperature," *ASME Journal of Heat Transfer,* Vol. 39, May 1971, pp. 136–145.

[6]Visser, W., "A Finite Element Method for the Determination of Non-Stationary Temperature Distribution and Thermal Deformations," *Proceedings of the Conference on Matrix Methods in Structural Mechanics,* Air Force Institute of Technology, Wright Patterson Air Force Base, Dayton, OH, 1965, pp. 925–943.

[7]Bruch, J.C. and Zyvoloski, G., "Transient Two-Dimensional Heat Conduction Problems Solved by the Finite Element Method," *International Journal for Numerical Methods in Engineering,* Vol. 8, No. 3, 1974, pp. 481–494.

[8]Bell, G.E., "A Method for Treating Boundary Singularities in Time-Dependent Problems," TR/8, Department of Mathematics, Brunel University of Uxbridge, Middlesex, England, 1972.

[9]Rizzo, F.J. and Shippy, D.J., "A Method of Solution for Certain Problems of Transient Heat Conduction," *AIAA Journal,* Vol. 11, Nov. 1970, p. 2004–2009.

[10]Liggett, J.A. and Liu, P.L.-F., "Unsteady Flow in Confined Aquifers—A Comparison of Two Boundary Integral Methods," *Water Resources Research,* Vol. 15, Aug. 1979, pp. 861–866.

[11]Rources, V. and Alarcon, E., "Transient Heat Conduction Problems Using B.I.E.M.," *Computers and Structures,* Vol. 16, No. 6, 1983, pp. 717–730.

[12]Cheng, A.H-D. and Liggett, J.A., "Boundary Integral Equation Method for Linear Porous-Elasticity with Applications to Soil Consolution," *International Journal for Numerical Methods in Engineering,* Vol. 20, Feb. 1984, pp. 255–278.

[13]Bhattacharya, M.C., "An Explicit Conditionally Stable Finite Difference Equation for Heat Conduction Problems," *International Journal for Numerical Methods in Engineering,* Vol. 21, Feb. 1985, pp. 239–265.

[14]Lick, W., "Improved Difference Approximations to the Heat Equation," *International Journal for Numerical Methods in Engineering,* Vol. 21, Nov. 1985, pp. 1957–1969.

[15]Myers, G.E., "Analytical Methods in Conduction Heat Transfer," McGraw-Hill, New York, 1971, pp. 263–416.

[16]Myers, G.E., "Finite-Difference/Finite-Element Class Notes," Report, University of Wisconsin, Madison, WI, 1978.

[17]Myers, G.E., "The Critical Time Step for Finite-Element Solutions to Two-Dimensional Heat-Conduction Transients," *ASME Journal of Heat Transfer,* Vol. 100, Feb. 1978, pp. 120–127.

[18]Ceylan, H.T., "Long-Time Solutions to Heat-Conduction Transients with Time-Dependent Inputs," Ph.D Thesis, Department of Mechanical Engineering, University of Wisconsin-Madison, WI, 1979.

[19]Dubner, W.M. and Abate, J., "Numerical Inversion of Laplace Transforms by Relating Them to the Finite Fourier Cosine Transform," *Journal of the ACM,* Vol. 15, Jan. 1968, pp. 115–123.

[20]Durbin, F., "Numerical Inversion of Laplace Transforms: Efficient Improvement to Dubner and Abate's Method," *The Computer Journal,* Vol. 17, June 1973, pp. 371–376.

[21]Segerlind, L.J., "Applied Finite Element Analysis," Wiley, New York, 1976, pp. 138–161.

[22]Huebner, K.H., "The Finite Element Method for Engineers," Wiley, New York, 1975, pp. 106–122.